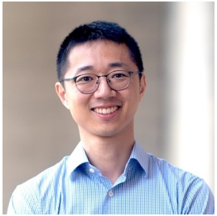
Hidden convexity, optimization, and algorithms on rotation matrices

Mathematics of Operations Research, 2024
ISMP, Montreal Canada





Akshay Ramachandran
Univ. of Amsterdam

Kevin Shu
CalTech

Alex L. Wang
Purdue

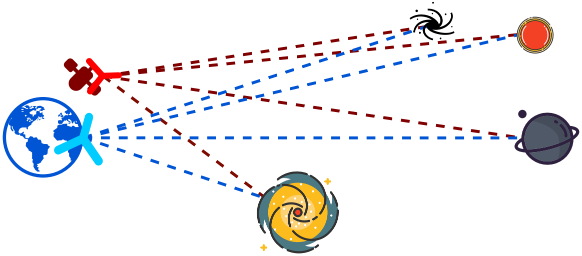
Outline

- Motivation: Wahba's problem
  - Variants of Wahba's problem with additional information
- Hidden convexity $\longrightarrow$ Exact convex relaxation
- Main results: Some variants of Wahba's problem have hidden convexity
- Picture proof of one of the main results

1. Wahba's problem
2. Hidden convexity
3. Some picture proofs
4. Conclusion

Wahba's problem I

- A satellite in space needs to determine its rotation (relative to an observer)
- There are k far away landmarks that it can see
- In the satellite's frame of reference, these landmarks are at $\{u_i\} \in \mathbb{R}^3$
- In the observer's frame of reference, these landmarks are at $\{v_i\} \in \mathbb{R}^3$



Wahba's problem II

- The set of rotations – Special Orthogonal Group

$$\mathrm{SO}(n) := \left\{ X \in \mathbb{R}^{n\times n} : \begin{array}{l} X^\intercal X = I \\ \det(X) = 1 \end{array} \right\}$$

- Wahba's problem: Find

$$\underset{X\in\mathrm{SO}(3)}{\arg\min} \sum_{i=1}^{n} \|Xv_i - u_i\|^2 = \underset{X\in\mathrm{SO}(3)}{\arg\max} \langle A, X\rangle$$

  where $A = \sum_{i=1}^{k} u_i v_i^\intercal$

- Solution can be found in closed form given an SVD of A

See Wahba [1965] and Farrell et al. [1966]

Wahba's problem with additional constraints

- Variant 1: Within some fixed angle of a prior rotation $\hat{X}$

$$\max_{X \in \mathrm{SO}(3)} \left\{ \langle A, X \rangle : \left\langle \hat{X}, X \right\rangle \geq \alpha \right\}$$

- Variant 2: Additional high-fidelity observations $\{w_i\}, \{z_i\}$

$$\max_{X \in \mathrm{SO}(3)} \{ \langle A, X \rangle : \langle z_i w_i^{\intercal}, X \rangle \geq \alpha, \ \forall i \}$$

- Can we algorithmically solve these problems?

1 Wahba's problem

2 Hidden convexity

3 Some picture proofs

4 Conclusion

- General problem

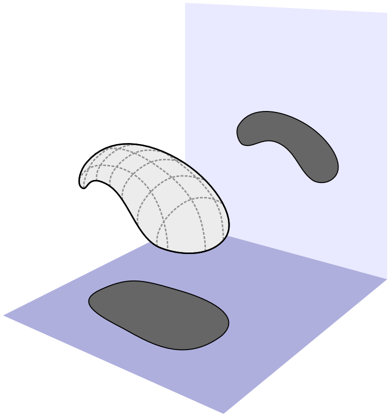
$$\sup_{X \in \mathrm{SO}(n)} \{\langle A, X\rangle : \mathcal{B}(X) \in \mathcal{C}\}$$

where $A \in \mathbb{R}^{n\times n}$, $\mathcal{B} : \mathbb{R}^{n\times n} \to \mathbb{R}^m$ is linear, $\mathcal{C} \subseteq \mathbb{R}^m$ is a “simple” convex set

- Let $\mathcal{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{1+m}$ by stacking A and $\mathcal{B}$

$$\mathcal{L}(X) := \begin{pmatrix} \langle A, X\rangle \\ \mathcal{B}(X) \end{pmatrix}$$

- Hidden convexity holds if $\mathcal{L}(\mathrm{SO}(n))$ is convex
- This is the linear image of a nonconvex set



Hidden convexity II

- Hidden convexity implies

$$\mathcal{L}(\mathrm{SO}(n)) = \mathrm{conv}(\mathcal{L}(\mathrm{SO}(n))) = \mathcal{L}(\mathrm{conv}(\mathrm{SO}(n)))$$

  and

$$\sup_{X \in \mathrm{SO}(n)} \{\langle A, X \rangle : \mathcal{B}(X) \in \mathcal{C}\} = \sup_{X \in \mathrm{conv}(\mathrm{SO}(n))} \{\langle A, X \rangle : \mathcal{B}(X) \in \mathcal{C}\}$$

- $\mathrm{conv}(\mathrm{SO}(n))$ is SDP-representable, thus RHS is a semidefinite program [1]
- Question: For what $\mathcal{L}$ is $\mathcal{L}(\mathrm{SO}(n))$ convex?

[1] Saunderson et al. [2015]

Main results

Theorem

Let $n \geq 3$. Suppose $\mathcal{L} : \mathrm{SO}(n) \to \mathbb{R}^2$ is linear. Then, $\mathcal{L}(\mathrm{SO}(n))$ is convex.

This is a hidden convexity result for variant 1 of Wahba's problem.

Theorem

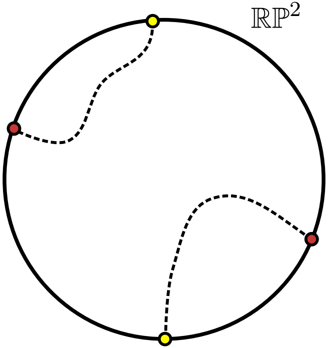
Let $\mathcal{L} : \mathrm{SO}(n) \to \mathbb{R}^{\binom{n}{2}}$ map an $n \times n$ matrix to its strictly upper triangular entries. Then, $\mathcal{L}(\mathrm{SO}(n))$ is convex.

This is a hidden convexity result for variant 2 of Wahba's problem (with at most $n - 1$ high fidelity observations).

1. Wahba's problem
2. Hidden convexity
3. Some picture proofs
4. Conclusion

Algebraic topology basics

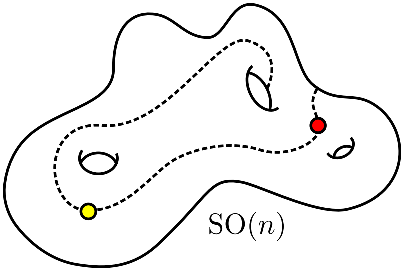
- The fundamental group is a group whose elements are "loops" and the group operation is concatenation
- Two loops are equivalent if one can be continuously deformed to the other
- A loop is "contractible" if it can be continuously deformed to a point
- **Fact**: The fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$
- Given any loop, the doubled-up loop is contractible
- For $n \geq 3$, same is true for $\mathrm{SO}(n)$

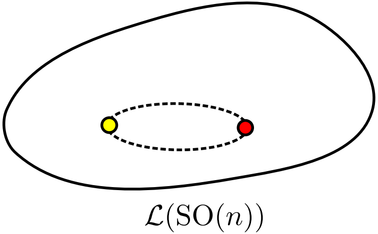


Theorem. Suppose $n \geq 3$ and $\mathcal{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^2$.
Then, $\mathcal{L}(\mathrm{SO}(n))$ is convex.

Proof sketch.

- Let $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ in the image
- There is a "loop" from X to Y and back in $\mathrm{SO}(n)$
- The image of this loop in $\mathbb{R}^2$ is an ellipse
- Double the loop
- Contract the doubled-up loop to a point
- Line between $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ is contained in $\mathcal{L}(\mathrm{SO}(n))$ $\square$





1. Wahba's problem
2. Hidden convexity
3. Some picture proofs
4. Conclusion

Conclusion

- New hidden convexity results for constrained optimization over $\mathrm{SO}(n)$
- Applies to variants of Wahba's problem
- Additional results in paper:
  - Fast algorithms for solving these problems
  - Structural results regarding completions of $\mathrm{SO}(n)$ matrices
  - Maximality of our hidden convexity results
- See paper: `arXiv:2304.08596`
- Thank you for listening! Questions?

References I

Brickman, L. (1961). On the field of values of a matrix. *Proc. Amer. Math. Soc.*, 12(1):61–66.

Dines, L. L. (1941). On the mapping of quadratic forms. *Bull. Amer. Math. Soc.*, 47(6):494–498.

Farrell, J., Stuelpnagel, J., Wessner, R., Velman, J., and Brook, J. (1966). A least squares estimate of satellite attitude (grace wahba). *SIAM review*, 8(3):384–386.

Saunderson, J., Parrilo, P., and Willsky, A. S. (2015). Semidefinite descriptions of the convex hull of rotation matrices. *SIAM J. Optim.*, 25(3):1314–1343.

Wahba, G. (1965). A least squares estimate of satellite attitude. *SIAM Rev.*, 7(3):409–409.

Quadratic convexity theorems

- There is a quadratic map $Q : \mathbb{R}^{2^{n-1}} \to \mathbb{R}^{n\times n}$ and a subset of the unit sphere $\mathrm{spin}(n)$ so that

$$Q(\mathrm{spin}(n)) = \mathrm{SO}(n)$$

- By "quadratic" we mean there exists $A_{ij} \in \mathbb{S}^{2^{n-1}}$ such that $Q(x)_{ij} = x^\mathsf{T} A_{ij} x$
- Suppose $\mathcal{L}(\mathrm{SO}(n))$ is convex. Then,

$$\mathcal{L}(\mathrm{SO}(n)) = (\mathcal{L}\circ Q)(\mathrm{spin}(n)) \subseteq (\mathcal{L}\circ Q)(\mathbf{S}^{2^{n-1}-1}) \overset{*}{\subseteq} \mathrm{conv}(\mathcal{L}(\mathrm{SO}(n))) = \mathcal{L}(\mathrm{SO}(n))$$

- Thus, $\mathcal{L}\circ Q$ is a entry-wise quadratic map such that $(\mathcal{L}\circ Q)(\mathbf{S}^{2^{n-1}-1})$ is convex
- New variants of Brickman's Theorem

Related: Saunderson et al. [2015], Dines [1941], Brickman [1961]